Subsampling Cumulative Covariance Estimator

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Abstract

In this paper subsampling *Cumulative Covariance* (CC) estimator is investigated specially on the optimal numbers of its subgrids. Although Hayashi and Yoshida (2005) proposed CC estimator to solve nonsynchronous bias problem under non-noise assumption, recent studies suggest that subsampling of CC estimator works for microstructure noise. However, it is also recognized that optimizing the numbers of subgrids is not straightforward. We provide a framework to select the subgrids by minimizing finite sample mean squared error. Some Monte Carlo studies are carried out to confirm the theory.

Keywords: High frequency data; Realized covariance; Nonsynchronous observation; Microstructure noise; Cumulative covariance estimator; Subsampling

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1 Introduction

Realized estimators for volatility or cross volatility have been getting more useful as the availability of ultra high frequency data increases. However, the market microstructure noise has prevented researchers from using highest frequency data such as transaction or quote data. Researchers had been compelled to apply moderate data frequency at which the effects of the noise are considered to be negligible. For instance, Andersen et al. (2003), one of the most influential works among realized volatility literatures, employed 15 minutes returns of foreign exchange rates. It was natural for researchers to desire a rigorous theory to select the appropriate data frequency. By such motivation Bandi and Russell (2005) provided the optimal frequency for realized volatility based on finite sample mean squared error (MSE). As for asymptotic theory, Zhang et al. (2005) provided a consistent realized estimator that is constructed by two different frequencies data, therefore called *Two Scale* (TS) estimator. Subsampling technique plays a central role in the consistency of TS estimator. Lower frequency data mitigate effect from noise while the averaging of realized volatilities constructed from lower frequency data contributes to the consistency of the resulting estimator. The *realized kernel* developed by Barndorff-Nielsen et al. (2008) unified several estimators including TS estimator and presented discussions on the asymptotic efficiency for various kernels. To get more efficiency, Barndorff-Nielsen et al. (2007) examined subsampling of realized kernels.

In contrast to volatility estimation, cross-volatility has not been studied enough. Nonsynchronicity between two different observations is one of the most crucial reasons why we cannot directly apply the theories on realized estimators of volatility to those of cross-volatility. Hayashi and Yoshida (2005) proposed an unbiased and consistent covariance estimator for asynchronous observation in the absence of the noise. It is called *Cumulative Covariance* estimator (CC) and solves nonsynchronous bias problem under the assumption of uncorrelated noise. Griffin and Oomen (2006) derived how many observations should be used or discarded for CC estimator under a bit restricted situation where volatilities are constant, noise is independent, and prices are observed in Poisson random manner. Voev and Lunde (2007) examined CC estimator under more general assumptions and suggested that subsampling CC could improve the MSE. Thus they also provided a rough criterion for the optimal number of subgrids. Choosing the optimal subgrids is the most inevitable but difficult step for practical use of subsampling CC. However Voev and Lunde (2007)'s criterion needs too many simplifications.

In order to provided a rigorous and practical procedure for optimal number of subgrids for subsampling CC, we apply finite sample MSE of *weighted realized covariance* (WRC) which was derived by Kanatani (2007). We provide a framework to minimize the MSE of subsampling CC by choosing optimal numbers of subgrids for two different series of original observations. Through the Monte Carlo study, we find that by applying the proposed procedure for optimal subgrids subsampling CC significantly outperforms other existing estimators.

The rest of the paper is organized as follows. In Section 2, we present assumptions on true price process and microstructure noise. In Section 3, we review finite sample MSE of WRC and present how to evaluate it. Section 4 we apply it to subsampling CC. In Section 5, we examine subsampling CC through some Monte Carlo studies and Section 6 concludes the paper.

2 Assumptions

We consider multi-dimensional vector of logarithmic asset price $p(t)$ for $t \geq 0$. Without loss of generality we set the dimension of p as 2. We assume that p is a continuous stochastic volatility semimartingale ($\mathcal{SVSM}^c$) with zero drift.[1]

$$p(t) = \int_0^t \Sigma(u) dz(u),$$

where Σ has elements that are all cadlag and z is a vector standard Brownian motion. We set the drift vector as 0 for the purpose of simplification.[2] The instantaneous or spot covariance matrix is defined by

$$\Omega(t) \equiv \Sigma(t)\Sigma(t)',$$

Instantaneous cross-volatility between 1st and 2nd asset is denoted as the $(1,2)$ or $(2,1)$ element of Ω:

$$\omega_{12}(t) = \sigma_{11}(t)\,\sigma_{21}(t) + \sigma_{12}(t)\,\sigma_{22}(t)\,.$$

Our target is not spot covariance but integrated cross volatility over $[0, T]$:

$$IC \equiv \int_0^T \omega_{12}(t)\,dt.$$

For the case of $i = j$, we call $IV_i \equiv \int_0^T \omega_{ii}(t)dt = \int_0^T (\sigma_{i1}(t)^2 + \sigma_{i2}(t)^2)dt$ the integrated variance. For estimation of the integrated volatility matrix, the following quadratic variation formula is a theoretical basis for using sum of outer product of return vector. If all assets are synchronously observed at the same time points

$$0 = t_0 < t_1 < \cdots < t_N = T,$$

and

$$\lim_{N\to\infty} \max_n (t_n - t_{n-1}) = 0$$

[1]See Barndorff-Nielsen and Shephard (2004) for the $\mathcal{SVSM}^c$.

[2]This simplification is acceptable in the context of high frequency data because the martingale component swamps the drift term over short time intervals.

then

$$p-\lim_{N\to\infty}\sum_{n=1}^{N}(p(t_n)-p(t_{n-1}))(p(t_n)-p(t_{n-1}))' = \int_0^T \Omega(t)dt.$$

See e. g. Barndorff-Nielsen and Shephard (2004).

However, if we consider ultra high frequency data such as transaction or quote data, each ith asset price must be nonsynchronously observed at different time points

$$0 = t_{0_i} < t_{1_i} < \cdots < t_{N_i} = T.$$

Usually in practice, nonsynchronous data are transformed into synchronous data by data manipulation scheme such as previous-tick interpolation. However, such manipulation should cause the nonsynchronous bias on realized covariance estimator which is known as *Epps effect*. See e. g. Kanatani and Renò (2007) or Zhang (2006) etc. Hayashi and Yoshida (2005) proposed a new estimator which can solve the nonsynchronous bias problem in the absence of the observation error.

More crucially, the efficient prices are considered to be contaminated by market microstructure noise. Define observed logarithmic asset price:

$$p_i^o(t_{n_i}) \equiv p_i(t_{n_i}) + e_i(t_{n_i}),$$

where $e(t)$ is independent with any other variables and $E(e(t)) = 0, V(e_i(t)) = \sigma_i^2$.[3] Define observed return as

$$r_i^o(t_{n_i}) \equiv r_i(t_{n_i}) + u_i(t_{n_i}),$$

where $r_i^o(t_{n_i}) \equiv p_i^o(t_{n_i}) - p_i^o(t_{n_i-1})$, $r_i(t_{n_i}) \equiv p_i(t_{n_i}) - p_i(t_{n_i-1})$, and $u_i(t_{n_i}) \equiv e(t_{n_i}) - e(t_{n_i-1})$. Notice that $r_i(t_{n_i})$ and $u_i(t_{n_i})$ have the zero-mean but

[3]Voev and Lunde (2007) employ more general structure of noises and examine its effect to Hayashi and Yoshida (2005)'s estimator. Ubukata and Oya (2007) propose how to test correlation of noises between different assets.

have different variances of $\int_{t_{n_i-1}}^{t_{n_i}} \omega_{ii}(t)dt$ and $2\sigma_i^2$ which are at orders of $O(t_{n_i} - t_{n_i-1})$ and $O(1)$, respectively. Therefore, under high frequency situation where $t_{n_i} - t_{n_i-1}$ is sufficiently small, the true return $r_i(t_{n_i})$ is overwhelmed by the noise term $u_i(t_{n_i})$.

Since we concentrate on measuring the *ex post* cross volatility from a given observation and do not make any hypothesis on the structure of the underlying probability space, we can consider $\Sigma(t)$ and t_{n_i} as deterministic functions.

3 MSE of weighted realized covariance

In this section we review the *weighted realized covariance* (WRC) which was proposed in Kanatani (2004). WRC is the general form of realized estimators nesting low frequency RV, subsampling methods, TS estimator, Fourier estimator[4], and Realized kernels, therefore, it enables us to unify the discussion on all of them. WRC is defined as follows.

$$WRC = (r_1^o)' W r_2^o,$$

where $r_i^o = (r_i^o(0), ..., r_i^o(t_{n_i}), ..., r_i^o(T))'$ and W is $N_1 \times N_2$ matrix. WRC is decomposed into

$$WRC = r_1' W r_2 + r_1' W u_2 + u_1' W r_2 + u_1' W u_2,$$

where $r_i = (r_i(t_{1_i}), ..., r_i(t_{n_i}), ..., r_i(T))'$ and $u_i = (u_i(t_{1_i}), ..., u_i(t_{n_i}), ..., u_i(T))'$. Note that the first term represents the WRC in the absence of the noise. For convenience, we introduce some notations. We denote the elements in W as

$$\begin{cases} w_{n_1 n_2}^{diag} & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ w_{n_1 n_2}^{off} & \text{otherwise.} \end{cases}$$

[4]Fourier estimator was proposed by Malliavin and Mancino (2002) as a new estimator based on Fourier analysis. Therefore it is considered to be very different from realized estimators. However, Kanatani (2004) showed that the Fourier estimator is a special form of realized estimator or WRC.

We denote the piecewise integrated covariance as

$$IC_{n_1 n_2} = \begin{cases} \displaystyle\int_{\max\{t_{n_1-1}, t_{n_2-1}\}}^{\min\{t_{n_1}, t_{n_2}\}} \omega_{12}(t)dt & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

and also denote the piecewise integrated variance as $IV_{n_i} = \int_{t_{n_i-1}}^{t_{n_i}} \omega_{ii}(t)dt$. Using the properties of independent increment of Brownian motion and uncorrelated noise, the bias of WRC is calculated as

$$E[WRC - IC] = \sum_{n_1, n_2} (w^{diag}_{n_1 n_2} - 1) IC_{n_1 n_2}. \tag{3.1}$$

Therefore, WRC is unbiased if $w^{diag}_{n_1 n_2} = 1$. In the special case of $w^{diag}_{n_1 n_2} = 1$ and $w^{off}_{n_1 n_2} = 0$, WRC is equivalent with Cumulative Covariance estimator proposed by Hayashi and Yoshida (2005). We examine subsampling CC in the later section. Since the independent noise does not affects the expectation of WRC, the bias comes from nonsynchronicity only. However, the noise does affect the MSE, which is calculated as

$$\begin{aligned} MSE &= E[WRC - IC]^2 \\ &= \underbrace{E[r_1' W r_2 - IC]^2}_{A} + \underbrace{E[r_1' W u_2]^2}_{B} + \underbrace{E[u_1' W r_2]^2}_{C} + \underbrace{E[u_1' W u_2]^2}_{D} \end{aligned} \tag{3.2}$$

where

$$A = \sum_{n_1, n_2} \left(w^{diag}_{n_1 n_2} IC_{n_1 n_2}\right)^2 + \sum_{n_1, n_2} w^2_{n_1 n_2} IV_{n_1} IV_{n_2} + \left\{ \sum_{n_1, n_2} (w^{diag}_{n_1 n_2} - 1) IC_{n_1 n_2} \right\}^2$$

$$B = 2\sigma_2^2 \sum_{n_1, n_2} w_{n_1 n_2} (w_{n_1 n_2} - w_{n_1 n_2 - 1}) IV_{n_1}$$

$$C = 2\sigma_1^2 \sum_{n_1, n_2} w_{n_1 n_2} (w_{n_1 n_2} - w_{n_1 - 1 n_2}) IV_{n_2}$$

$$D = \sigma_1^2 \sigma_2^2 \sum_{n_1, n_2} w_{n_1 n_2} \left\{4 w_{n_1 n_2} + 2 w_{n_1 - 1 n_2 - 1} + 2 w_{n_1 - 1 n_2 + 1} - 4 (w_{n_1 - 1 n_2} + w_{n_1 n_2 - 1})\right\}$$

$w_{n_1 n_2} = 0$ if $n_i \leq 0$ or $n_i \geq N_i$.

See Kanatani (2007) for the detail of calculation.

Since the WRC is a bit too general to minimize the MSE, we need to select a specific form of weight function. For the purpose of simple MSE evaluation, we limit our discussion within unbiased estimators in other words, we set $w_{n_1n_2}^{diag} = 1$. This allows us to avoid evaluating the bias and $\sum \left(w_{n_1n_2}^{diag} IC_{n_1n_2}\right)^2$. If $w_{n_1n_2}^{diag} = 1$, the bias is zero, and $\sum \left(w_{n_1n_2}^{diag} IC_{n_1n_2}\right)^2$ is unknown, however, constant. So we do not need to evaluate piecewise integrated covariance $IC_{n_1n_2}$ that is difficult to estimate.

In order to evaluate the MSE, we still need the variance of noise σ_i^2 and the piecewise integrated volatility IV_{n_i}. It is difficult to estimate every piecewise integrated volatility IV_{n_i} as well as $IC_{n_1n_2}$. To avoid evaluating IV_{n_i}, we impose the assumption: "Volatility does not change so much over $[0, T]$." This assumption is seen in Bandi and Russell (2006). Under this assumption, the following approximation is valid.[5]

$$IV_{n_i} \approx \frac{IV_i \Delta t_{n_i}}{T}, \tag{3.3}$$

where $\Delta t_{n_i} = t_{n_i} - t_{n_i - 1}$. As for estimations of σ_i^2 and IV_i, there are several established methods, see e.g. Bandi and Russell (2005), Zhang et al. (2005), etc. Estimations of them are not the purpose of this paper, we treat them as known parameters.

Now the minimization of the finite sample MSE of WRC reduces to

$$\min_{\theta}(A' + B' + C' + D) \tag{3.4}$$

[5]Bandi and Russell (2006) use the approximation $IV_{n_i} \approx IV_i/N_i$ to derive optimal frequency base on finite sample MSE of subsampling estimator. However, such approximation implies that "time difference does not change so much" is also assumed. So we use less restricted approximation (3.3) since our present purpose is not to derive optimal frequency explicitly.

where

$$
\begin{aligned}
A' &= T^{-2} IV_1 IV_2 \sum w_{n_1 n_2}^2 \Delta t_{n_1} \Delta t_{n_2} \\
B' &= 2T^{-1} IV_1 \sigma_2^2 \sum w_{n_1 n_2} (w_{n_1 n_2} - w_{n_1 n_2 - 1}) \Delta t_{n_1} \\
C' &= 2T^{-1} IV_2 \sigma_1^2 \sum w_{n_1 n_2} (w_{n_1 n_2} - w_{n_1 - 1 n_2}) \Delta t_{n_2} \\
w_{n_1 n_2} &= \begin{cases} 1 & \text{if } (t_{n_1 - 1}, t_{n_1}] \cap (t_{n_2 - 1}, t_{n_2}] \neq \emptyset \\ f(t_{n_1}, t_{n_2}; \theta) & \text{otherwise.} \end{cases}
\end{aligned}
$$

In the next section, we see subsampling CC as a concrete example of WRC.

4 Subsampling of Cumulative Covariance estimator

Hayashi and Yoshida (2005) proposed unbiased estimator for nonsynchronous true observations, which is named Cumulative Covariance (CC) estimator:

$$
CC = \sum_{n_1, n_2} r_1^o(t_{t_{n_1}}) r_2^o(t_{t_{n_2}}) 1\{(t_{n_1 - 1}, t_{n_1}] \cap (t_{n_2 - 1}, t_{n_2}] \neq \emptyset\} \tag{4.1}
$$

where $1\{\cdot\}$ denotes the indicator function. The estimator is trivially written in the WRC expression with the following weight.

$$
w_{n_1 n_2} = \begin{cases} 1 & \text{if } (t_{n_1 - 1}, t_{n_1}] \cap (t_{n_2 - 1}, t_{n_2}] \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}
$$

To mitigate the effect of the noise, Griffin and Oomen (2006) studied lower frequency version of CC estimator:

$$
\begin{aligned}
CC(k) = \sum_{n_1, n_2} &\{p_1^o(t_{kt_{n_1}}) - p_1^o(t_{k(t_{n_1} - 1)})\}\{p_2^o(t_{kt_{n_2}}) - p_2^o(t_{k(t_{n_2} - 1)})\} \\
&\times 1\{(t_{k(n_1 - 1)}, t_{kn_1}] \cap (t_{k(n_2 - 1)}, t_{kn_2} \neq \emptyset\}.
\end{aligned}
$$

where k is a positive integer. In matrix expression, the estimator has the weight:

$$w_{n_1n_2} = \begin{cases} 1 & \text{if } (t_{k(n_1-1)}, t_{kn_1}] \cap (t_{k(n_2-1)}, t_{kn_2}] \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

Griffin and Oomen (2006) calculate the MSE to optimize k under the condition of constant volatilities and Poisson random sampling. However we can select optimal k by minimizing (3.2) under more general settings. Griffin and Oomen (2006) also suggested that subsampling version of CC could get much more efficiency. Let positive integers K_1 and K_2 be numbers of subgrid for first and second base asset respectively. Subsampling CC is defined as

$$SCC(K_1, K_2) = \frac{1}{K_1K_2} \sum_{k_1=1}^{K_1} \sum_{k_2=1}^{K_2} CC(k_1, k_2, K_1, K_2), \tag{4.2}$$

where

$$\begin{aligned} &CC(k_1, k_2, K_1, K_2) = \\ &\sum_{n_1,n_2} \{p_1^o(t_{K_1t_{n_1}} + k_1) - p_1^o(t_{K_1(t_{n_1}-1)} + k_1)\}\{p_2^o(t_{K_2t_{n_2}} + k_2) - p_2^o(t_{K_2(t_{n_2}-1)} + k_2)\} \\ &\qquad \times 1\{(t_{K_1(n_1-1)+k_1}, t_{K_1n_1+k_1}] \cap (t_{K_2(n_2-1)+k_2}, t_{K_2n_2+k_2} \neq \emptyset\}. \end{aligned}$$

Voev and Lunde (2007) examined it for the special case of one subgrid ($K_1 = K_2$) and provided a rough criterion for the optimal subgrid. However their criterion can be calculated only under strong conditions.

Now we consider more flexible selection of subgrids under more general assumptions. For that purpose, we must get the matrix expression of (4.2). Then it is sufficient that we apply the theory of WRC's MSE. Let

$$K_1^- = \begin{cases} K_1 & \text{if } \frac{N-k_1}{K_1} = \lfloor \frac{N-k_1}{K_1} \rfloor \\ N_1 - \lfloor \frac{N-k_1}{K_1} \rfloor K_1 - k_1 & \text{otherwise} \end{cases}$$

Define the weight matrix for subsampling CC as

$$
{}_{K_1K_2}D_{k_1k_2} = \begin{pmatrix}
1_{k_1k_2} & 0_{k_1K_2} & \cdots & \cdots & 0_{k_1K_2} & 0_{k_1K_2^-} \\
0_{K_1k_2} & 1_{K_1K_2} & 0_{K_1K_2} & \cdots & 0_{K_1K_2} & 0_{K_1K_2^-} \\
\vdots & 0_{K_1K_2} & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0_{K_1K_2} & \vdots \\
0_{K_1k_2} & 0_{K_1K_2} & \cdots & 0_{K_1K_2} & 1_{K_1K_2} & 0_{K_1K_2^-} \\
0_{K_1^-k_2} & 0_{K_1^-K_2} & \cdots & \cdots & 0_{K_1^-K_2} & 1_{K_1^-K_2^-}
\end{pmatrix}
$$

where 1_{mn} and 0_{mn} are $m \times n$ matrix whose all elements are unity and zero respectively. Then we get the matrix expression of subsampling CC:

$$
\begin{aligned}
SCC(K_1, K_2) &= \frac{1}{K_1K_2} \sum_{k_1=1}^{K_1} \sum_{k_2=1}^{K_2} (r_1^o)' {}_{K_1K_2}D_{k_1k_2} r_2^o \\
&= (r_1^o)' \underbrace{\left(\frac{1}{K_1K_2} \sum_{k_1=1}^{K_1} \sum_{k_2=1}^{K_2} {}_{K_1K_2}D_{k_1k_2} \right)}_{W^{SCC}} r_2^o
\end{aligned}
$$

In practice we should pactition the matrix W^{SCC} and vectors r_1^o and r_2^o if they are too large to compute. Note also that $SCC(K_1, K_2)$ is unbiased by definition.

5 Monte Carlo study

In this section, we performed Monte Carlo simulations to confirm the validity of our theory. In our simulation, efficient price process is generated by

$$
\begin{pmatrix} dp_1(t) \\ dp_2(t) \end{pmatrix} = \begin{pmatrix} \sigma_{11}(t) & 0 \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \begin{pmatrix} dz_1(t) \\ dz_2(t) \end{pmatrix}, \ 0 \le t \le T
$$
$$
d\sigma_{ij}(t) = \kappa \left(\theta - \sigma_{ij}(t) \right) dt + \gamma dz_{ij}(t), \ i, j = 1, 2.
$$

where $\kappa = 0.1$, $\theta = 1$, $\gamma = 0.1$, $T = 1$(day). We generate a proxy of the process with a time step $\Delta = 1/60 \times 60 \times 4.5$(one second precision for

Japanese stock exchanges). Time differences are drawn from an exponential distribution:

$$F\left(t_{n_i} - t_{n_i-1}\right) = 1 - \exp\left\{-\lambda_i\left(t_{n_i} - t_{n_i-1}\right)\right\},\ i = 1, 2$$

where $F(\cdot)$ denotes a cumulative distribution function, $\lambda_i = 1/60\Delta$ $(i = 1, 2)$. It means that the average time difference is 60 seconds for each asset. At the each time point, the efficient price is observed with independent noise: $e_1(t_{n_1}) \sim NID(0, \sigma_1^2)$, $e_2(t_{n_2}) \sim NID(0, \sigma_2^2)$ where $(\sigma_1^2, \sigma_2^2) = (0.025, 0.05)$.

For comparison with the subsampling CC, we selected three estimators which got the best performances in the similar experiment by Kanatani (2007). One of them is the WRC estimator with error function weight:

$$w_{n_1 n_2} = \begin{cases} 1 & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ \exp\left(-\left(\frac{t_{n_1} - t_{n_2}}{h}\right)^2\right) & \text{otherwise.} \end{cases}$$

where $h > 0$. The other two have compact supports. Their weight functions are

$$w_{n_1 n_2} = \begin{cases} 1 & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] \neq \emptyset \\ k\left(\frac{|t_{n_1} - t_{n_2}|}{H}\right) & \text{if } (t_{n_1-1}, t_{n_1}] \cap (t_{n_2-1}, t_{n_2}] = \emptyset \text{ and } |t_{n_1} - t_{n_2}| < H \\ 0 & \text{otherwise.} \end{cases}$$

where $H > 0$. If $f(x) = 1 - x$ and $f(x) = (1 - \cos\pi(1 - x)^2)/2$, they are called Bartlett and Modified Turkey-Hanning kernel respectively.

By controlling h or H, the effect from noise can be mitigated. We also presented performances of low frequency CC ($CC(k)$), simple CC ($CC(1)$), and product of daily return $((p_1^o(T) - p_1^o(0))(p_2^o(T) - p_2^o(0)))$. The optimal parameters were selected by solving the minimization (3.4) with true values of σ_i^2 and IV_i.

We performed 500 daily replications to compute sample root MSE and

Table 1: Sample RMSE and MAE with average of optimal parameter

	RMSE	MAE	Ave. of optimal parameter
Daily Return	1.590	1.194	—
$CC(1)$	0.941	0.737	—
$CC(k^*)$	0.693	0.551	$\bar{k}^* = 8.220$
Error Function	0.388	0.308	$\bar{h}^* = 0.028$
Bartlett	0.390	0.310	$\bar{H}^* = 0.052$
Mod. Turkey-Hanning	0.386	0.306	$\bar{H}^* = 0.083$
$SCC(K_1^*, K_2^*)$	0.347	0.281	$(\bar{K_1}^*, \bar{K_2}^*) = (5.23, 4.74)$

($\lambda_1\Delta$,$\lambda_2\Delta$)=(1/60,1/60),(σ_1^2,σ_2^2)=(0.025,0.05). 500 replications

mean absolute error (MAE):

$$RMSE = \sqrt{\frac{1}{500}\sum_{r=1}^{500}\left(estimate^{(r)} - IC^{(r)}\right)^2},$$

$$MAE = \frac{1}{500}\sum_{r=1}^{500}|estimate^{(r)} - IC^{(r)}|.$$

Table 1 shows sample RMSE and MAE. Averages of selected parameters are also reported. Subsampling CC outperforms all the other estimators in terms of both RMSE and MAE. It is natural because subsampling CC is optimized by controlling two parameters while the others are controlled by at most one parameter.

We also performed another experiment under different simulation settings. We set parameters as $(\sigma_1^2, \sigma_2^2) = (0.05, 0.025)$ and $(\lambda_1\Delta, \lambda_2\Delta) = (1/30, 1/60)$ to observe more asymmetric situation: double noise variance and half duration for the 1st asset while half noise variance and double duration for 2nd asset. It means that 1st asset is influenced from noise much more while 2nd much less. Selected numbers of subgrids in Table 2 show subsampling CC can capture the asymmetric influence from noise.

Table 2: Sample RMSE and MAE with average of optimal parameter

	RMSE	MAE	Ave. of optimal parameter
Daily Return	1.720	1.120	—
$CC(1)$	0.964	0.764	—
$CC(k^*)$	0.754	0.595	$\bar{k}^* = 6.232$
Error Function	0.471	0.374	$\bar{h}^* = 0.034$
Bartlett	0.461	0.367	$\bar{H}^* = 0.058$
Mod. Turkey-Hanning	0.469	0.373	$\bar{H}^* = 0.094$
$SCC(K_1^*, K_2^*)$	0.421	0.334	$(\bar{K_1}^*, \bar{K_2}^*) = (10.6, 2.36)$

($\lambda_1\Delta$,$\lambda_2\Delta$)=(1/30,1/120),(σ_1^2,σ_2^2)=(0.05,0.025). 500 replications

6 Concluding remarks

In this paper we study subsampling CC and how to select the optimal numbers of subgrids for them. By applying the evaluation procedure for the finite sample MSE of WRC, we can decide the optimal numbers of subgrids. Monte Carlo study shows that the optimally subsampled CC improves MSE significantly compared to existing estimators. Relaxing assumptions on noise structure and studying subsampling kernel estimator are now under development.

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