Convexity and d-revised link-convexity of restricted games on some intersecting family

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Abstract We consider some restricted games which are generalization of the games with graph-communication structure. We propose a result of in-essential coalitions structure of some intersecting convex games. Also we show a relationship between link-convexity and d-revised link-convexity for games on some intersecting family.
Key words: Intersecting family, Link-convexity, Restricted games

1 Introduction

In this paper we assume that only a certain family of groups have a sufficient internal structure to operate as a coalition. Such a group of players are denoted as a feasible coalition. Myerson [9] considered cooperation under explicit communication restrictions imposed by communication network among the players.

For the class of game with communication structure represented by an undirected graph, a solution called the average tree solution has been studied recently. For example Herings et al. [6] proposed the condition of link-convexity under which the average tree solution belongs to the core for the class of arbitrary undirected graph games. On the other hand, Igarashi and Yamamoto [7] revised definition of link-convexity to guarantee that the average tree solution is an element of the core for the class of arbitrary undirected graph games. Also they [8] showed that link-convexity of [6] coincides with d-revised link-convexity for the class of cycle-complete graph games.

On the contrary, feasible coalition can not always be represented by an undirected graph. As a natural result, more general models for the games with graph-communication structure have been studied. We also consider some restricted games which are generalization of the graph-communication structure. In section 3 we introduce quasi-intersecting convexity and propose a result of in-essential coalitins structure. In section 4 and 5 we study a generalization of Igarashi and Yamamoto [7] results on convexity and d-revised link-convexity of games with cycle-complete graph-communication structure.

2 Definitions and preliminaries

Let $N = \{1, 2, \ldots, n\}$ be a set of players. A *coalition* is a nonempty subset of N and a *characteristic function* $v : 2^N \to \mathbf{R}$ assigns to each coalition S its *value* $v(S)$. We denote (N, v) an n-person game in characteristic function form where v is a real-valued function on 2^N and $v(\emptyset) = 0$. In addition, we may call v game on N. A transferable utility cooperative

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game with communication structure is represented by (N, v, L). The edge set L is a set of communication links $L = \{\{i,j\} \mid i \neq j,\ i,j \in N\}$, i.e., $\{i,j\} \in L$ if and only if there exists communication between i and j. The undirected graph $G = (N, L)$ consists of the vertex set N and the edge set L is called communication graph of the game (N, v, L).

A graph G' is called a *subgraph of* G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$. For a vertex subset $S \subseteq V(G)$, the subgraph $G[S]$ of G whose vertex set is S and whose edge set consists of the edges of G joining vertices of S is called the subgraph of G induced by S.

A *path* in $G = (N, L)$ is a sequence $P = (i_1, \cdots, i_m)$ of different vertices i_k $(k = 1, \cdots, m)$ such that $\{i_k, i_{k+1}\} \in L$ $(k = 1, \ldots, m-1)$. The vertices i_1 and i_m are *linked* by P. The number of edges of a path is its *length*. A nonempty graph $G[S]$ is called *connected* if any two of its vertices are linked by a path. A graph is called *complete* if any two of its nodes are connected by an edge. A sequence of vertices $(i_1, \cdots, i_m, i_1)$ is called a *cycle* if $(i_1, \cdots, i_m)$ is a path whose length is at least 3 and $\{i_m, i_1\} \in L$. A graph $G = (N, L)$ is said to be *cycle-complete* if there is a cycle $(i_1, \cdots, i_k, i_1)$ in the graph then $G[\{i_1, \cdots, i_k\}]$ is complete.

In the game (N, v, L) with communication graph G, coalition S is feasible if and only if $G[S]$ is connected. For any $S \in 2^N$, let $C^L(S)$ denote the collection of all feasible coalitions in the graph $G[S]$. A feasible coalition K is a *component* of S if and only if $K \in C^L(N)$ is a maximal subset of S. The graph restricted game (N, v^L) with communication structure (N, v, L) is defined by

$$v^L(S) = \sum_{K \in \widehat{C}^L(S)} v(K)$$

where $\widehat{C}^L(S)$ is the set of all components of S.

In this paper we consider more general models for the games with graph-communication structure. A restricted game is a triple $(N, v, \mathcal{F})$, where $\mathcal{F}$ is a nonempty collection of subsets of N called *feasible coalitions*. We assume $N \in \mathcal{F}$ without loss of generality. A pair $(N, \mathcal{F})$ is called a *feasible coalition system*. For any $S \in 2^N$, $C^{\mathcal{F}}(S) = \{K \mid K \subseteq S \text{ and } K \in \mathcal{F}\}$. K is a *component* of S if and only if $K \in \mathcal{F}$ is a maximal subset of S. The collection of all components of S is denoted by $\widehat{C}^{\mathcal{F}}(S)$.

A characteristic function v is superadditive if

$$v(S) + v(T) \leq v(S \cup T)$$

for all disjoint pair of $S, T \in 2^N$. A characteristic function v is convex if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for all $S, T \in 2^N$.

For any $S \subseteq N$ and $x \in \mathbf{R}^n$ define

$$x(S) = \sum_{i \in S} x_i$$

where $x(\emptyset) = 0$. The core of a game (N, v) is defined by

$$\mathrm{Core}(v) = \{x \mid x \in \mathbf{R}^n, x(N) = v(N), \forall S \subseteq N \colon x(S) \geq v(S)\}.$$

The core of a restricted game $(N, v, \mathcal{F})$ is defined by

$$\mathrm{Core}(N, v, \mathcal{F}) = \{x \mid x \in \mathbf{R}^n, x(N) = v(N), \forall S \in \mathcal{F} \colon x(S) \geq v(S)\}.$$

3 Quasi-intersecting convex games

A family $\mathcal{F}$ of subsets of N is called an *intersecting family* if for each intersecting pair of $S, T \in \mathcal{F}$ (i.e., $S \cap T \neq \emptyset$) we have $S \cup T \in \mathcal{F}$ and $S \cap T \in \mathcal{F}$. A set-function $v : \mathcal{F} \to R$ is called an *intersecting-supermodular function on the intersecting family* $\mathcal{F}$ if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

holds for each intersecting pair of $S, T \in \mathcal{F}$. If $\mathcal{F}$ is an intersecting family and v is an intersecting-supermodular function on the intersecting family $\mathcal{F}$ we call $(N, v, \mathcal{F})$ an *intersecting convex game*.

A collection $\Pi = \{A_1, ..., A_k\} \subseteq \mathcal{F}$ is called an $\mathcal{F}$-partition of A if $A = A_1 \cup \cdots \cup A_k$ where $A_1, ..., A_k \in \mathcal{F}$ are pairwise nonempty disjoint feasible coalitions. We denote by $P_{\mathcal{F}}(A)$ the family of all $\mathcal{F}$-partition of A. An $\mathcal{F}$*-subpartition* Π of a set A is a set of nonempty disjoint subsets of A consisting of feasible coalitions. We refer to an element A_i of an $\mathcal{F}$-partition Π as a *block* of Π. The collection of all $\mathcal{F}$-subpartitions of N is denoted by $SP_{\mathcal{F}}(N)$.

For a set function $g : \mathcal{F} \to R$ and an $\mathcal{F}$-subpartition $\Pi \in SP_{\mathcal{F}}(N)$, we define

$$\overline{g}(\Pi) = \sum_{X \in \Pi} g(X).$$

If $\mathcal{F}$ is an intersecting family, then we define a partial order $\succeq$ on $SP_{\mathcal{F}}(N)$ by defining $\Pi_1 \succeq \Pi_2$ if and only if each block of Π_2 is contained in some block of Π_1. The least (greatest) element of $SP_{\mathcal{F}}(N)$ above (below) Π_1 and Π_2 in the partially ordered set $SP_{\mathcal{F}}(N)$ is denoted by $\Pi_1 \vee \Pi_2$ ($\Pi_1 \wedge \Pi_2$). We should notice that $\Pi_1 \wedge \Pi_2$ does not always exist for two arbitrary $\mathcal{F}$-subpartitions Π_1, Π_2. However, by defining $\{\emptyset\} \preceq \Pi \in SP_{\mathcal{F}}(N)$ and $\{\emptyset\} \preceq \{\emptyset\}$, $SP_{\mathcal{F}}(N) \cup \{\{\emptyset\}\}$ forms a lattice with $\preceq$.

Definition 3.1: Let $\mathcal{F}_{\mathcal{D}}$ consist of all those subsets $A \subseteq N$ which can be written as

$$A = \bigcup_{A_i \in \Pi} A_i,$$

where $\Pi \in P_{\mathcal{F}}(A)$. □

Using the similar argument as in the proof of Theorem3.3 [10], we have the following theorem.

Theorem 3.2 (cf. [10]): *Let* $A, B \in \mathcal{F}_{\mathcal{D}}$ *such that* $B \subseteq A$. *Let* $\Pi_1 \in P_{\mathcal{F}}(A)$ *and* $\Pi_2 \in P_{\mathcal{F}}(B)$. *For an intersecting-supermodular function* $v : \mathcal{F} \to R$, *if* $\overline{v}(\Pi_1) = \max_{\Pi \in P_{\mathcal{F}}(A)} \overline{v}(\Pi)$ *and* $\overline{v}(\Pi_2) = \max_{\Pi \in P_{\mathcal{F}}(B)} \overline{v}(\Pi)$, *then* $\overline{v}(\Pi_1 \vee \Pi_2) = \max_{\Pi \in P_{\mathcal{F}}(A)} \overline{v}(\Pi)$ *and* $\overline{v}(\Pi_1 \wedge \Pi_2) = \max_{\Pi \in P_{\mathcal{F}}(B)} \overline{v}(\Pi)$. □

Theorem 3.2 is a generalization of the following Narayanan's Theorem.

Theorem 3.3 ([11] Theorem 3.5): *Let* $\Pi_1, \Pi_2 \in P_N$, *where* P_N *is the collection of all partitions of* N. *For any supermodular function* f *on* 2^N *with* $f(\emptyset) = 0$, *if* $\overline{f}(\Pi_1) = \overline{f}(\Pi_2) = \max_{\Pi \in P_N} \overline{f}(\Pi)$, *then* $\overline{f}(\Pi_1 \vee \Pi_2) = \overline{f}(\Pi_1 \wedge \Pi_2) = \max_{\Pi \in P_N} \overline{f}(\Pi)$. □

Corollary 3.4: *Let* $A, B \in \mathcal{F}_{\mathcal{D}}$ *such that* $B \subseteq A$ *and* v *be an intersecting-supermodular function.*
(1) *For any* $\Pi_1 \in \arg\max_{\Pi \in P_{\mathcal{F}}(A)} \overline{v}(\Pi)$ *there exists* $\Pi_2 \in \arg\max_{\Pi \in P_{\mathcal{F}}(B)} \overline{v}(\Pi)$ *with* $\Pi_2 \preceq \Pi_1$.
(2) *For any* $\Pi'_2 \in \arg\max_{\Pi \in P_{\mathcal{F}}(B)} \overline{v}(\Pi)$ *there exists* $\Pi'_1 \in \arg\max_{\Pi \in P_{\mathcal{F}}(A)} \overline{v}(\Pi)$ *with* $\Pi'_2 \preceq \Pi'_1$. □

Igarashi and Yamamoto [7] introduced the concept of convexity of graph game (N, v, L).

Definition 3.5: *(N, v, L) is convex if*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for all $S, T \in C^L(N)$ satisfying $S \cup T \in C^L(N)$ and $S \cap T \in C^L(N) \cup \{\emptyset\}$. □

In conjunction with Definition 3.5, we introduce special class of intersecting convex game.

Definition 3.6: *$(N, v, \mathcal{F})$ is quasi-intersecting convex if $\mathcal{F}$ is an intersecting family and*

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

for all $S, T \in \mathcal{F}$ satisfying $S \cup T \in \mathcal{F}$. □

In a convex game (N, v) a coalition S is called *inessencial* if it has a proper partition $\Pi = \{S_1, \ldots, S_t\} \in P_{\mathcal{F}}(S) - \{\{S\}\}$ such that $v(S) = \overline{v}(\Pi)$. Coalitions which are not inessential are called *essential*. For an intersecting convex game $(N, v, \mathcal{F})$ we denote by $\mathcal{E}_{\mathcal{F}}$ the collection of its nonempty essential coalitions. The collection $\mathcal{E}_{\mathcal{F}}$ gives a description of the core as follows:

$$\mathrm{Core}(N, v, \mathcal{F}) = \{x \mid x \in \mathbf{R}^n, x(N) = v(N), \forall S \in \mathcal{E}_{\mathcal{F}} : x(S) \geq v(S)\}.$$

A partition Π of S is called an *$\mathcal{E}_{\mathcal{F}}$-partition* of S if each block of Π is nonempty essential coallition. We note that if v is superadditive and coalition S is inessential, then $v(S) = \sum_{S_i \in \Pi} v(S_i)$ for some $\mathcal{E}_{\mathcal{F}}$-partition Π of S.

Theorem 3.7: *Let $(N, v, \mathcal{F})$ be a quasi-intersecting convex game. If S is an inessential coalition, then there exists a unique $\mathcal{E}_{\mathcal{F}}$-partition Π^* of S such that $v(S) = \overline{v}(\Pi^*)$.*

Proof: The proof is by contradiction. Suppose there exist two different $\mathcal{E}_{\mathcal{F}}$-partition $\Pi_1, \Pi_2 \in P_{\mathcal{F}}(S) - \{\{S\}\}$ such that

$$v(S) = \overline{v}(\Pi_1) = \overline{v}(\Pi_2).$$

Using properties of the quasi-intersecting convexity of v, we have

$$\overline{v}(\Pi_1) = \overline{v}(\Pi_2) = \max_{\Pi \in P_{\mathcal{F}}(S) - \{\{S\}\}} \overline{v}(\Pi).$$

Hence, to apply Theorem 3.2 we obtain

$$v(S) = \overline{v}(\Pi_1 \wedge \Pi_2).$$

However, since $\Pi_1 \wedge \Pi_2 \preceq \Pi_1$, if $\Pi_1 \wedge \Pi_2 = \Pi_1$, then at least one block of Π_2 is inessential. On the other hand, if $\Pi_1 \wedge \Pi_2 \prec \Pi_1$, then at least one block of Π_1 is inessential. □

4 Convexity of $\mathcal{F}$-restricted games

A *partition system* is a feasible coalition system $(N, \mathcal{F})$ such that for all $S \in 2^N$, $\widehat{C}^{\mathcal{F}}(S)$ is an $\mathcal{F}$-partition of S. If $(N, \mathcal{F})$ is a partition system then $\mathcal{F}_{\mathcal{D}} \cup \{\emptyset\} = 2^N$.

Definition 4.1: Let $(N, v, \mathcal{F})$ be a restiricted game and $(N, \mathcal{F})$ a partition system. The $\mathcal{F}$-restricted game $(N, v^{\mathcal{F}})$ is defined by

$$v^{\mathcal{F}}(S) = \sum_{K \in \widehat{C}^{\mathcal{F}}(S)} v(K)$$

where $v^{\mathcal{F}} : 2^N \to \mathbf{R}$. □

ALGABA, BILBAO and LOPEZ [1] proved the following theorem.

Theorem 4.2 ([1] Theorem 3): *A feasible coalition system $(N, \mathcal{F})$ is a partition system if and only if $\mathcal{F}$ satisfies the following three conditions:*
(P1) $\emptyset \in \mathcal{F}$
(P2) $\{i\} \in \mathcal{F}$ *for all* $i \in N$
(P3) $S \cup T \in \mathcal{F}$ *for all intersecting pair of* $S, T \in \mathcal{F}$. □

For a set function $v : \mathcal{F} \to \mathbf{R}$, the set function $\widehat{v} : \mathcal{F}_{\mathcal{D}} \cup \{\emptyset\} \to \mathbf{R}$ is defined by

$$\widehat{v}(A) = \begin{cases} \max_{\Pi \in P_{\mathcal{F}}(A)} \overline{v}(\Pi) & (A \in \mathcal{F}_{\mathcal{D}}) \\ 0 & (A = \emptyset). \end{cases}$$

If $\widehat{C}^{\mathcal{F}}(S)$ is a partition of S and v is superadditive, then

$$\sum_{K \in \widehat{C}^{\mathcal{F}}(S)} v(K) = \widehat{v}(S). \tag{4.1}$$

Fujishige [4] and Faigle [3] proved the following lemma.

Lemma 4.3 ([4] Lemma 5.1, [3] Lemma 11): *Let $(N, v, \mathcal{F})$ be an intersecting convex game. Then*

$$\widehat{v}(S) + \widehat{v}(T) \leq \widehat{v}(S \cup T) + \widehat{v}(S \cap T)$$

for all $S, T \in \mathcal{F}_{\mathcal{D}} \cup \{\emptyset\}$. □

Theorem 4.4: *Let $\mathcal{F}$ be an intersecting family that satisfies conditions (P1) and (P2) of Theorem 4.2. $(N, v, \mathcal{F})$ is a quasi-intersecting convex game if and only if the restricted game $(N, v^{\mathcal{F}})$ is convex.*

Proof: First we suppose that $(N, v^{\mathcal{F}})$ is convex. For all intersecting pair of $S, T \in \mathcal{F}$, we have

$$v(S) + v(T) = v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) \leq v^{\mathcal{F}}(S \cup T) + v^{\mathcal{F}}(S \cap T) = v(S \cup T) + v(S \cap T)$$

because $S \cup T \in \mathcal{F}$ and $S \cap T \in \mathcal{F}$. On the other hand, for all $S, T \in \mathcal{F}$ satisfying $S \cap T = \emptyset$ and $S \cup T \in \mathcal{F}$, we obtain

$$v(S) + v(T) = v^{\mathcal{F}}(S) + v^{\mathcal{F}}(T) \leq v^{\mathcal{F}}(S \cup T) = v(S \cup T).$$

Hence, $(N, v, \mathcal{F})$ is a quasi-intersecting convex game.

Next we suppose that $(N, v, \mathcal{F})$ is a quasi-intersecting convex game. From (4.1) we have

$$v^{\mathcal{F}}(S) = \sum_{K \in \widehat{C}^{\mathcal{F}}(S)} v(K) = \widehat{v}(S)$$

for all $S \in \mathcal{F}$. Since $(N, v, \mathcal{F})$ is also an intersecting convex game, Lemma 4.3 implies that $(N, v^{\mathcal{F}})$ is convex. □

Igarashi and Yamamoto [7] showed that convexity of (N, v, L) is a necessary and sufficient condition for convexity of (N, v^L) when the underlying graph is cycle-complete ([7] Theorem 4.7). They also show that if $G = (N, L)$ is a cycle-complete graph, then for all $S, T \in C^L(N)$ with $S \cap T \neq \emptyset$ satisfy $S \cap T, S \cup T \in C^L(N)$. Thus, Theorem 4.4 implies that if G is a cycle-complete graph, then (N, v, L) is a convex game if and only if the graph restricted game (N, v^L) is convex.

5 D-revised link-convexity on some union stable and intersecting family

A family $\mathcal{F}$ of subsets of N is called a *union stable family* if for for all intersecting pair of $S, T \in \mathcal{F}$ we have $S \cup T \in \mathcal{F}$. A feasible coalition system $(N, \mathcal{F})$ is called a *union stable system* if $\mathcal{F}$ is a union stable family. For example, $C^L(N)$ of a graph game (N, v, L) is a union stable family. Also a particular case of union stable family is intersecting family.

Definition 5.1: *$(N, v, \mathcal{F})$ is link-convex if $\mathcal{F}$ is a union stable family and*

$$v(S) + v(T) \leq v(S \cup T) + \sum_{K \in \widehat{C}^{\mathcal{F}}(S \cap T)} v(K)$$

for all $S, T \in \mathcal{F}$ that satisfy
(LC1)$S \setminus T, T \setminus S, (S \setminus T) \cup (T \setminus S) \in \mathcal{F}$
(LC2)$N \setminus S \in \mathcal{F}$ *or* $N \setminus T \in \mathcal{F}$. □

Note that if $\mathcal{F}$ is a union stable family and $S, T \in \mathcal{F}$ satisfy (LC1) then $S \cup T \in \mathcal{F}$ even if $S \cap T = \emptyset$.

Theorem 5.2: *Let $\mathcal{F}$ be an intersecting family that satisfies conditions (P1) and (P2) of Theorem 4.2. If $(N, v, \mathcal{F})$ is a quasi-intersecting convex game, then $(N, v, \mathcal{F})$ is link-convex.*

Proof: If $S, T \in \mathcal{F}$ then $\widehat{C}^F(S \cap T) = \{S \cap T\}$, hence $\sum_{K \in \widehat{C}^F(S \cap T)} v(K) = v(S \cap T)$. □

Theorem 5.2 is a generalization of Igarashi and Yamamoto [7] Theorem 4.8.

Definition 5.3: *$(N, v, \mathcal{F})$ is revised link-convex if $\mathcal{F}$ is a union stable family and*

$$v(S) + v(T) \leq v(S \cup T) + \sum_{K \in \widehat{C}^{\mathcal{F}}(S \cap T)} v(K)$$

for all $S, T \in \mathcal{F}$ that satisfy
(RL1)$S \setminus T, S \cup T \in \mathcal{F}$
(RL2)$N \setminus S \in \mathcal{F}$ *or* $N \setminus T \in \mathcal{F}$. □

If $(N, v^{\mathcal{F}})$ is a convex game, then $(N, v, \mathcal{F})$ is revised link-convex because $v^{\mathcal{F}}(S \cup T) = v(S \cup T)$ and $v^{\mathcal{F}}(S \cap T) = \sum_{K \in \widehat{C}^{\mathcal{F}}(S \cap T)} v(K)$.

Definition 5.4: *$(N, v, \mathcal{F})$ is d-revised link-convex if $\mathcal{F}$ is a union stable family and*

$$v(S) + v(T) \leq v(S \cup T) + \sum_{K \in \widehat{C}^{\mathcal{F}}(S \cap T)} v(K)$$

for all $S, T \in \mathcal{F}$ that satisfy
(dRL1)$S \setminus T, S \cup T \in \mathcal{F}$
(dRL2)$N \setminus S \in \mathcal{F}$
(dRL3)*there exists $j \in T \setminus S$ such that $(S \setminus T) \cup \{j\} \in \mathcal{F}$.* □

Definition 5.5: *A family $\mathcal{F}$ is called link-union stable if $\mathcal{F}$ is union stable and for all nonempty disjoint pair of $S, T \in \mathcal{F}$ with $S \cup T \in \mathcal{F}$, there exists $j \in T$ such that $S \cup \{j\} \in \mathcal{F}$.* □

$C^L(N)$ of a graph game (N, v, L) is also a link-union stable family.

Theorem 5.6: *Let $\mathcal{F}$ be a link-union stable family and an intersecting family. Then $(N, v, \mathcal{F})$ is link-convex if and only if $(N, v, \mathcal{F})$ is d-revised link-convex.*

Proof: Suppose $S, T \in \mathcal{F}$ satisfy the conditions (dRL1), (dRL2), and (dRL3). We will show that the pair S, T also satisfy the conditions (LC1) and (LC2). Obviously S, T satisfy (LC2) because (dRL2) implies $N \setminus S \in \mathcal{F}$. Hence, we only show that S, T satisfy the conditin (LC1).

If $S \cap T = \emptyset$, then $S \setminus T = S \in \mathcal{F}$ and $T \setminus S = T \in \mathcal{F}$. From (dRL1) we obtain $(S \setminus T) \cup (T \setminus S) = S \cup T \in \mathcal{F}$. Therefore S, T satisfy (LC1).

If $S \cap T \neq \emptyset$, then $T \cap (N \setminus S) \neq \emptyset$ because $T \setminus S \neq \emptyset$ by (dRL3). Since $\mathcal{F}$ is an intersecting family, $T \in \mathcal{F}$ and (dRL2) imply $T \setminus S = T \cap (N \setminus S) \in \mathcal{F}$. Also $S \setminus T \in \mathcal{F}$ by (dRL1). From (dRL3) we obtain $(S \setminus T) \cup (T \setminus S) = ((S \setminus T) \cup \{j\}) \cup (T \setminus S) \in \mathcal{F}$. Hence S, T satisfy (LC1).

Conversly, suppose $S, T \in \mathcal{F}$ satisfy the conditions (LC1) and (LC2). We will show that the pair S, T also satisfy the conditions (dRL1), (dRL2), and (dRL3). Without loss of generality we assume $N \setminus S \in \mathcal{F}$. Then S, T satisfy (dRL1) by (LC1) and (dRL2) by (LC2). Moreover, since $\mathcal{F}$ is link-union stable family, S, T satisfy (dRL3) by (LC1). □

A particular case of a link-union stable family and an intersecting family is $C^L(N)$ of a cycle-complete graph game. Hence, for cycle-complete graph game (N, v, L) is link-convex if and only if (N, v, L) is d-revised link-convex (Igarashi and Yamamoto [8]).

Definition 5.7: *$(N, v, \mathcal{F})$ is superadditive if $\mathcal{F}$ is a union stable family and*

$$v(S) + v(T) \leq v(S \cup T)$$

for all disjoint pair of $S, T \in \mathcal{F}$ that satisfy $S \cup T \in \mathcal{F}$. □

Definition 5.8: *A family $\mathcal{F}$ is called unique link-union stable if $\mathcal{F}$ is union stable and for all nonempty disjoint pair of $S, T \in \mathcal{F}$ with $S \cup T \in \mathcal{F}$, there exists only one element $j \in T$ such that $S \cup \{j\} \in \mathcal{F}$.* □

Example 5.9: *Let $N = \{1, 2, 3, 4\}$ be a set of players. Then the following family*

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3, 4\}, \{1, 2, 3, 4\}\}$$

is a unique link-union stable. □

Theorem 5.10: *Let $\mathcal{F}$ be a unique link-union stable family and an intersecting family. Then $(N, v, \mathcal{F})$ is link-convex if for every $S \in \mathcal{F}$ it holds that*

$$v(S) + v(T) \le v(S \cup T)$$

for all $T \in \widehat{C}^{\mathcal{F}}(N \setminus S)$ satisfying $S \cup T \in \mathcal{F}$.

Proof: Let $S, T \in \mathcal{F}$ satisfy the conditions (LC1) and (LC2). We suppose $N \setminus T \in \mathcal{F}$ without loss of generality.

We first show that $S \cap T = \emptyset$. Assume the contrary, letting $S \cap T \neq \emptyset$. Then $S \cap T \in \mathcal{F}$ because $\mathcal{F}$ is an intersecting family. Since $S \setminus T, S \in \mathcal{F}$ and $\mathcal{F}$ is a unique link-union stable family, there exists $j_1 \in S \cap T$ such that

$$(S \setminus T) \cup \{j_1\} \in \mathcal{F}.$$

Moreover, by (LC1) and $\mathcal{F}$'s link-union stability, there exists $j_2 \in T \setminus S$ such that

$$(S \setminus T) \cup \{j_2\} \in \mathcal{F}.$$

However, there exists only one element $j \in T$ such that

$$(S \setminus T) \cup \{j\} \in \mathcal{F}$$

because $S \cup T \in \mathcal{F}$ and $\mathcal{F}$ is unique link-union stable. This contradicts the fact that $j_1 \neq j_2$ and $j_1, j_2 \in T$.

Next we show that $T \in \widehat{C}^{\mathcal{F}}(N \setminus S)$. From $S \cap T = \emptyset$ we have $S \subseteq N \setminus T, T \subseteq N \setminus S$. From the conditions (LC1) and (LC2) we obtain $S \cup T, N \setminus T \in \mathcal{F}$. Since $S, T, S \cup T \in \mathcal{F}$ and $\mathcal{F}$ is a unique link-union stable family, there exists $j_3 \in S$ such that

$$T \cup \{j_3\} \in \mathcal{F}.$$

Suppose $T \notin \widehat{C}^{\mathcal{F}}(N \setminus S)$. Then there exists T' such that $T \subset T' \in \widehat{C}^{\mathcal{F}}(N \setminus S)$. Thus $T' \setminus T = T' \cap (N \setminus T) \in \mathcal{F}$ because $\mathcal{F}$ is an intersecting family. Since $T, T' \setminus T, T' \in \mathcal{F}$ and $\mathcal{F}$ is unique link-union stable, there exists $j_4 \in T' \setminus T$ such that

$$T \cup \{j_4\} \in \mathcal{F}.$$

However, since $T, N \setminus T, N \in \mathcal{F}$, there exists only one element $j \in N \setminus T$ such that

$$T \cup \{j\} \in \mathcal{F}.$$

This contradicts the fact that $j_3 \neq j_4$ and $j_3, j_4 \in N \setminus T$. □

Corollary 5.11: *Let $\mathcal{F}$ be a unique link-union stable family and an intersecting family. Then $(N, v, \mathcal{F})$ is d-revised link-convex if $(N, v, \mathcal{F})$ is superadditive.* □

$C^L(N)$ of a cycle-free graph game (N, v, L) is also a unique link-union stable family. Hence, from Theorem 5.10 we obtain the following corollary.

Corollary 5.12 (Herings et al. [6] Corollary 2): *A game with cycle-free communication structure (N, v, L) is link-convex if (N, v, L) is superadditive.* □

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