EXACT CALCULATION OF BIT ERROR RATES IN CHAOS COMMUNICATION SYSTEMS

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Abstract. A bit error rate is an important aspect in communication modeling and used to assess the performance. However, bit error rates that have been reported in earlier work were either based on Gaussian-based approximation or computational simulation and led inaccurate results. In order to assess the performance of communication systems, modulating methods and chaotic maps that is used to produce a spreading sequence, exact bit error rates are highly required. The paper investigates the calculation of an exact bit error rate for single-user chaos shift-keying (CSK) systems, and gives several comparisons that can not be obtained by the approach of Gaussian based approximation or computational simulation either. The use of an optimal decoder is also mentioned and a comparison of the optimal decoder and the so-called correlation decoder is given from the viewpoint of exact bit error rates.

1. Introduction

Because of explosive growth in personal communications, especially between mobile communications terminals, there is a need to provide effective communication systems in which many users are allowed to access simultaneously without interference and communications are secure. For security, each user's stream of signals (messages) should look like noise to the others and these streams should be independent each other to avoid interference. In conventional communication systems, trigonometric wave functions are used to disguise digital information as a "noise-like" sequence and many modulation schemes have been proposed. However, because of the periodicity of trigonometric wave functions the risk of interference in multiple user systems is not low. Chaos communication systems are supposed to be the alternative in which chaotic sequences are used instead of trigonometric wave functions and a wide variety of chaos communication systems have already been proposed, mainly as adaptions of conventional systems. Kennedy, Rovatti and Setti [4] is the first monograph in the area.

In any communication system, an estimate of bit (digital information) based on a received "noise-like" sequence at a receiver side is required and there is no error-free system due to the channel noise. The channel is the physical medium through which a "noise-like" sequence passes as it travels from a sender to a receiver and is corrupted by noise, usually assumed to be additive white Gaussian noise (AWGN). Therefore the probability of an estimation error, *bit error rate* (*BER*), is of interest. Even though it seems clear that an estimation method (decoder) that gives a low bit error rate is preferable, a *correlation decoder* has been widely used with little consideration about its exact bit error rate. It is a purpose of this paper to reconsider a correlation decoder from the viewpoint of maximum likelihood estimation as suggested in Schimming and Hasler [10], and to revisit the calculation of bit error rates from first statistical and dynamical principles to obtain exact bit error rates; as exemplification of the approach, the simplest of single-user chaos shift

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keying models is studied. The approach makes use of the implicit distributional behaviour of chaotic processes and the widely employed assumption of AWGN in the channel. The existing literature on the calculation of bit error rates for particular chaos-based models induces some concern where un-justifiable statistical manipulations and Gaussian assumptions may have been made. These can lead to inaccurate results relative to exact calculation; Lawrance and Balakrishna [6] give a simple chaos shift keying illustration where only a Gaussian assumption is at fault. Examples of earlier work which might be extended by exact calculations include Abel *et al* [1], Kolumban [5], Lipton and Dabke [8], Milanovic *et al* [9], Sushchik *et al* [11], and Tam *et al* [12]. A further advantage of seeking exact results is that they enable detailed and accurate comparison of bit error rates using different chaotic map generators, with identical moments for instance, and may be lead to identification of some optimal of generators.

From the viewpoint of maximum likelihood estimation, a correlation decoder is not always optimal. On the other hand, a likelihood optimal decoder is generally not as tractable as the correlation decoder. It is thus investigated whether the optimal decoder should be used considering its intractability.

2. Chaos Shift-keying Systems and a Likelihood Optimal Decoder

Communication systems involving the generation of chaotic sequence will be introduced by mathematically specifying and discussing simple versions of the so-called *chaos shift-keying* (CSK) systems. In these systems messages to be sent are in binary bit form, here with a bit being denoted by $b = \pm 1$. Theoretically, attention is focused on the transmission of a single bit. A digital bit b is sent to a receiver by being embedded in an N length spreading sequence $\{X_i\}$ which is generated by a chaotic map $\tau(z)$, $c \leq z \leq d$. The length N is called the *spreading factor*. The sequence is assumed to have been started with a random value from an invariant distribution of the map, thus the each element of the sequence has the common mean μ and the common variance ${\sigma_X}^2$. The embedded signal is of the form $\mu + b(X_i - \mu)$ and thus the bit is signified by either leaving the chaotic sequence unchanged as X_i $(b = 1)$, or reflecting it about its mean μ as $2\mu - X_i$, a form of chaotic modulation. To avoid the reflected values being out of range, the invariant distribution is assumed to be symmetric and so $\mu = (d-c)/2$. Modulated signals go through the transmission channel and are corrupted by AWGN, $\{\varepsilon_i\}$, with variance σ^2. Thus the received signal is of the form $R_i = \mu + b(X_i - \mu) + \varepsilon_i$. In *coherent* CSK systems, a receiver is assumed to have all information about a spreading sequence $\{X_i\}$. On the other hand, a receiver is allowed to have only corrupted information about $\{X_i\}$ in *non-coherent* CSK systems. We will explore the optimal decoder based on the standard statistical method of maximum likelihood estimation first in coherent CSK systems, then in non-coherent CSK systems. Each bit is assumed to be independent of previous bits.

2.1. An optimal decoder in coherent CSK systems.

In coherent CSK systems, a receiver has the data $\{r_i\}$ and $\{x_i\}$ as the sample of $\{R_i\}$ and $\{X_i\}$ respectively to estimate digital information b. The proper approach is through maximum likelihood, taking $b = \pm 1$ according to which value gives the highest likelihood of $(b|r, x)$ where $r = (r_1, r_2, \ldots, r_N)'$ and $x = (x_1, x_2, \ldots, x_N)'$. This is the approach set out by Schimming and Hasler [10] to be applied here. Since the conditional distribution of R given X is independent Gaussian and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N)'$ and X are independent, the joint likelihood of b based on (r, x) is

$$f_{R,X}(r, x|b) = f_X(x) f_\varepsilon(r - \mu - b(x - \mu)), \tag{1}$$

where $f_X(\cdot)$ and $f_\varepsilon(\cdot)$ are the marginal densities of X and ε respectively. Thus, the likelihood of b based on (r, x) is

$$\ell(b|r,x) = f_X(x)(\sqrt{2\pi}\sigma)^{-N} \exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(r_i - \mu - b(x_i - \mu))^2).$$

The bit b is estimated by $\hat{b} = +1$ if $\ell(+1|r,x)/\ell(-1|r,x) \geq 1$ and by $\hat{b} = -1$ if the ratio is less than one. This inequality is seen to be equivalent to

$$C(r,x) \stackrel{\text{def}}{=} \sum_{i=1}^{N}(x_i - \mu)(r_i - \mu) \geq 0. \tag{2}$$

$C(r, x)$ is the so-called *correlation decoder* even though it involves covariance, not correlation, between the chaotic segment and the received bit segment. Nevertheless, the widely used correlation decoder is optimal in coherent CSK systems. It is of interest to calculate the error probability of the estimation, the *bit error rate* (*BER*). The approximation based on the standard Gaussian assumption has been used as if it is an exact bit error rate for a long time with little consideration, see Kennedy, Rovatti and Setti [4], pp22-23. However, it will be seen in the next section that the approximation gives only a lower bound of exact bit error rates. The exact calculation of bit error rates of the optimal decoder, that is, correlation decoder in coherent CSK systems will be given in the next section.

2.2. **An optimal decoder in non-coherent systems.**

In non-coherent systems, a receiver is supposed to have only corrupted information about $\{X_i\}$. Suppose the data a receiver can use to estimate b is $\{r_i\}$ and $\{y_i\}$ where $\{y_i\}$ is a sample from random variable

$$Y_i = X_i + \eta_i,$$

and $\{\eta_i\}$ is AWGN with the same variance σ^2 as $\{\varepsilon_i\}$. This system can be considered as *differential chaos shift-keying* (DCSK) in which a received sequence $\{R_i\}$ is of the form,

$$R_i = \begin{cases} \mu + b(X_i - \mu) + \varepsilon_i & (i = 1, 2, \ldots, N/2) \\ X_{i-N/2} + \varepsilon_i & (i = N/2+1, N/2+2, \ldots, N), \end{cases}$$

and this is the only data a receiver can use. The likelihood of b is based on the joint density of $(R, Y|b)$. First the joint density of (X, Y, R) is required, and by generalising (1), this is seen to be

$$f_{X,Y,R}(x,y,r|b) = f_X(x) f_\eta(y - x) f_\varepsilon(r - \mu - b(x - \mu))$$

and with the AWGN assumptions becomes

$$f_X(x)(2\pi\sigma^2)^{-N} \exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(y_i - x_i)^2) \exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}((r_i - \mu) - b(x_i - \mu))^2). \tag{3}$$

Marginalizing out the unknown exact spreading values X, the likelihood of b is seen as

$$\ell(b|r,y) = \int_{(c,d)^N} (2\pi\sigma^2)^{-N} \exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{N}\{(y_i - x_i)^2 + ((x_i - \mu) - b(r_i - \mu))^2\}) f_X(x) dx. \tag{4}$$

The summation term is now simplified so as to isolate the terms in x as much as possible and uncover a covariance type term for y. It can be written

$$\sum_{i=1}^{N} \left[\frac{(y_i-\mu)^2}{2} + \frac{(r_i-\mu)^2}{2} - b(r_i-\mu)(y_i-\mu) \right.$$
$$\left. + \, 2\left\{(x_i-\mu) - \frac{(y_i-\mu)+b(r_i-\mu)}{2}\right\}^2 \right].$$

In order to simplify subsequent expressions, put

$$t_0(y,r) = \frac{1}{2}\sum_{i=1}^{N}\{(y_i-\mu)^2 + (r_i-\mu)^2\}$$

and

$$t(x,y,r,b) = \sum_{i=1}^{N}\left\{(x_i-\mu) - \frac{(y_i-\mu)+b(r_i-\mu)}{2}\right\}^2.$$

The likelihood in (4) can now be written as

$$(2\pi\sigma^2)^{-N}\exp[-\frac{1}{2\sigma^2}\{t_0(y,r) - b\textstyle\sum_{i=1}^{N}(r_i-\mu)(y_i-\mu)\}]E[\exp\{-\frac{t(X,y,r,b)}{\sigma^2}\}]$$

which gives the likelihood ratio as

$$\frac{\ell(+1|y,r)}{\ell(-1|y,r)} = \exp\left\{\frac{1}{\sigma^2}\textstyle\sum_{i=1}^{N}(r_i-\mu)(y_i-\mu)\right\}\frac{E[\exp\{-t(X,y,r,+1)/\sigma^2\}]}{E[\exp\{-t(X,y,r,-1)/\sigma^2\}]}. \quad (5)$$

Taking logarithm of the right-hand side of (5), we have

$$Opt(y,r) \stackrel{\text{def}}{=} \frac{1}{\sigma^2}\sum_{i=1}^{N}(r_i-\mu)(y_i-\mu)$$
$$+ \log E[\exp\{-t(X,y,r,+1)/\sigma^2\}] - \log E[\exp\{-t(X,y,r,-1)/\sigma^2\}] \quad (6)$$

The bit b is estimated by $\hat{b} = +1$ if $Opt(y,r) \geq 0$ and by $\hat{b} = -1$ if $Opt(y,r) < 0$. The first term of (6) can be considered as the so-called correlation decoder using the signal and the spreading sequences, and the two logarithm terms in (6) can be considered as a modulating factor using the same sequences. Therefore the correlation decoder is not optimal in non-coherent CSK systems while it is in coherent CSK systems. Because of the modulating factor, the picture is not totally as clear as might be hoped, following the wishes of Schimming and Hasler [10]; moreover the modulating factor may be difficult to calculate repeatedly. It is of interest to assess "how much better the optimal decoder is than the correlation decoder." The comparison should be based on exact bit error rates, but the exact bit error rate of the optimal decoder can not be calculated straightforwardly because of the complexity of the modulating factor. Thus the exact bit error rate of the correlation decoder is still of interest though it is not optimal in terms of maximum likelihood estimates. We will explore the exact bit error rate of the correlation decoder in non-coherent systems in the next section. A comparison between exact bit error rates of the correlation decoder and simulated bit error rates of the optimal decoder will also be given.

3. Exact Bit Error Rates of the Correlation Decoder

3.1. Bit error rates in coherent CSK systems.

As we have seen in the previous section, the correlation decoder (2) is optimal in coherent CSK systems and the bit error rate of the case $b = 1$ is sent is given as follows:

$$BER(N) = \Pr(C(R, X) < 0 | b = 1).$$

It is easily seen to be the same when $b = -1$ is sent, although this is not always the case. From the definition of R_i and (2) we have

$$\begin{aligned} BER(N) &= \Pr(\textstyle\sum_{i=1}^{N}(X_i - \mu)(X_i - \mu + \varepsilon_i) < 0) \\ &= \Pr(\textstyle\sum_{i=1}^{N}\varepsilon_i(X_i - \mu) < -\sum_{i=1}^{N}(X_i - \mu)^2), \end{aligned}$$

and by the double expectation theorem,

$$= E_X[\Pr(\textstyle\sum_{i=1}^{N}\varepsilon_i(X_i - \mu) < -\sum_{i=1}^{N}(X_i - \mu)^2 | X)]. \quad (7)$$

The conditional probability in (7) can be written as follows: since $\{X_i\}$ and $\{\varepsilon_i\}$ are independent,

$$\begin{aligned} &\Pr(\textstyle\sum_{i=1}^{N}\varepsilon_i(X_i - \mu) < -\sum_{i=1}^{N}(X_i - \mu)^2 | X = x) \\ &= \Pr(\textstyle\sum_{i=1}^{N}(x_i - \mu)\varepsilon_i < -\sum_{i=1}^{N}(x_i - \mu)^2). \end{aligned} \quad (8)$$

Note that the left-hand side in the probability (8) is a linear combination of AWGN $\{\varepsilon_i\}$, so becomes

$$\Phi(-\sqrt{\textstyle\sum_{i=1}^{N}(x_i - \mu)^2}/\sigma)$$

where $\Phi(\cdot)$ is the distribution function of a standard Gaussian random variable. We thus have

$$BER(N) = E[\Phi(-\sqrt{\textstyle\sum_{i=1}^{N}(X_i - \mu)^2}/\sigma)], \quad (9)$$

where expectation is taken over the spreading sequence $\{X_i\}$ as dependent random variables. When a spreading sequence is chaotic, (9) can be written as

$$BER(N) = E[\Phi(-\sqrt{\textstyle\sum_{i=1}^{N}(\tau^{i-1}(X_1) - \mu)^2}/\sigma)],$$

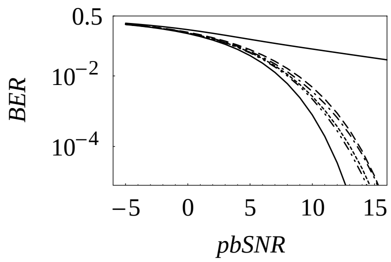
since $X_i = \tau^{i-1}(X_1)$. The result (9) holds for any type of spreading sequence assumed to have stationary probabilistic behaviour, and not necessarily chaotic ones, but only for Gaussian noise. By Jensen's inequality we have a lower bound of bit error rates as

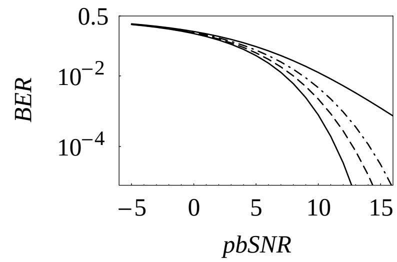
$$BER(N) = E[\Phi(-\sqrt{\textstyle\sum_{i=1}^{N}(X_i - \mu)^2}/\sigma)] \geq \Phi(-\sqrt{N}\sigma_X/\sigma). \quad (10)$$

It can be easily seen that in *binary phase shift-keying* (BPSK) systems in which a spreading sequence is independent *balanced binary*, that is,

$$X_i = \begin{cases} c & \text{with probability } 1/2 \\ d & \text{with probability } 1/2, \end{cases}$$

$BER(N)$ equals the lower bound $\Phi(-\sqrt{N}\sigma_X/\sigma)$. It should be noted that the lower bound (10) is the same as the widely used Gaussian approximation $erfc(\sqrt{\rho})/2$ where ρ is a standard error ratio of signal to noise, of spreading to noise in this case. This approximation has been used as an "analytical" bit error rate, but as we have seen it is only a lower bound in general. In Figure 1a and in subsequent similar figures, $BER(N)$ is plotted in log scale against $pbSNR$, the 'per bit signal to noise ratio' and measured in decibels as $10\log_{10}(N\sigma_X^2/\sigma^2)$. This definition thus incorporates a compensation for different spreading factors N and the use of the transmission channel. Figure 1a shows the beneficial effect of chaotic spreading; the curves $N = 2, 3, 4$ and 5 are bunched together much lower than the $N = 1$ case, although still a little way above the lower bound, but showing that the benefits of more extensive spreading will not be much greater - a conclusion which could not reliably be obtained from approximate calculations. Figure 1b shows reducing accuracy as the invariant distribution progressively departs from balanced binary in the order $\beta(\frac{1}{2}, \frac{1}{2})$, uniform and Gaussian. It can be conjectured that a chaotic map having the invariant distribution with smaller kurtosis gives a lower bit error rate as Lawrance and Balakrishna [6] suggested.





Left: **Figure 1a**. Bit error rate $BER(N)$ plotted against per bit signal to noise ratio $pbSNR$ in the case of coherent chaotic logistic spreading for spreading factors $N = 1$ (upper solid), $N = 2$ (dashed), $N = 3$ (dot dashed), $N = 4$ (dotted), $N = 5$ (dot dot dashed) and lower bound (lower solid).
Right: **Figure 1b**. Bit error rate $BER(N)$ plotted against per bit signal to noise ratio $pbSNR$ with spreading factor $N = 5$ for lower bound balanced binary spreading (lower solid), logistic map spreading (dashed), shift map spreading (dot dashed) and independent Gaussian spreading (upper solid).

3.2. **Bit error rates in non-coherent CSK system.**

Though the correlation decoder is not optimal in non-coherent CSK systems, it is still meaningful to calculate exact bit error rates of the correlation decoder in non-coherent CSK systems because of the difficulty calculating the modulating part in (6). Denoting the first term of (6) as $C(r, y)$ then $BER(N)$ is written as

$$\begin{aligned} BER(N) &= \Pr(C(R,Y) < 0 | b = 1) \\ &= \Pr(\sum_{i=1}^{N} (X_i - \mu + \eta_i)\varepsilon_i < -\sum_{i=1}^{N} (X_i - \mu)^2 - \sum_{i=1}^{N} (X_i - \mu)\eta_i). \end{aligned} \quad (11)$$

By working in terms of the independent random variables $\varepsilon_i + \eta_i$ and $\varepsilon_i - \eta_i$, (11) can be expressed exactly in terms of a non-central F distribution, Johnson *et al* [3], Chapter 30, with equal degrees of freedom (N, N) and non-central parameter $2\sum_{i=1}^{N}(X_i - \mu)^2/\sigma^2$. Denoting such a random variable by $F_{N,N}(2\sum_{i=1}^{N}(X_i - \mu)^2/\sigma^2)$, the required expression is

$$BER(N) = E[\Pr(F_{N,N}(2\textstyle\sum_{i=1}^{N}(X_i - \mu)^2/\sigma^2) < 1)] \quad (12)$$

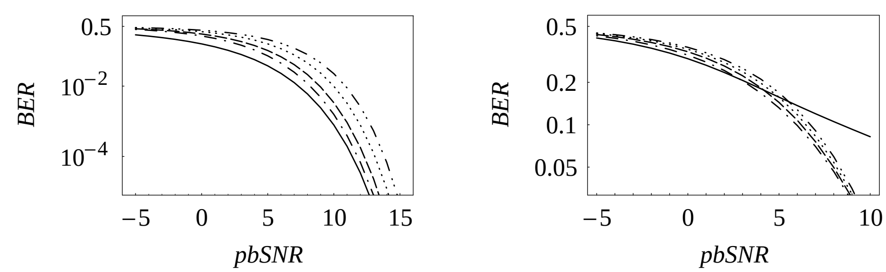
where the expectation is taken over the spreading sequence $\{X_t\}$ which may have chaotic dependency. Similarly to the coherent case, the result can be written as

$$BER(N) = E[\Pr(F_{N,N}(2\textstyle\sum_{i=1}^{N}(\tau^{i-1}(X_1) - \mu)^2/\sigma^2) < 1)]$$

when a spreading sequence is chaotic. The exact effects of noisy spreading can be obtained by comparing (12) and (9). The lower bound of (12) can be obtained by applying Jensen's inequality to the series expansion of the non-central F term and yields the computable result,

$$BER(N) \geq \Pr(F_{N,N}(2N\frac{\sigma_X^2}{\sigma^2}) < 1). \tag{13}$$

Similarly to the coherent case, the lower bound (13) equals the $BER(N)$ of non-coherent BPSK systems. In Figure 2a there are comparisons of the lower bounds (13) and (10) of the bit error rates in terms of *pbSNR*, for several spreading factors N. There is just one lower bound curve in the coherent case while the lower bound bit error rate with fixed *pbSNR* increases as N increases in the non-coherent case. The comparisons in Figure 2b show the existence of the optimal spreading factor for a constant *pbSNR* in non-coherent systems that does not exist in coherent systems. This might be due to the noise in spreading sequence at a receiver side.



Left: **Figure 2a**. Comparison of $BER(N)$ for the coherent and non-coherent lower bound, plotted against *pbSNR*. The solid curve applies to all spreading factors in the coherent case. In the non-coherent case, the lower bound curves are given for $N = 1$ (dot dashed), $N = 2$ (dashed), $N = 5$ (dotted) and $N = 10$ (dot dot dashed).
Right: **Figure 2b**. Bit error rate $BER(N)$ plotted against *pbSNR* in the case of non-coherent logistic spreading for spreading factors $N = 1$ (solid), $N = 2$ (dot dashed), $N = 3$ (dashed), $N = 4$ (dotted) and $N = 5$ (dot dot dashed).

It should be noted that if a signal to noise ratio, σ_X^2/σ^2, is fixed instead of fixed per bit signal to noise ratio $N\sigma_X^2/\sigma^2$, increasing N in (13) reduces the bit error rate lower bound but at slower pace than in the coherent case (10). The increase in bit error rate lower bound as N increases in non-coherent CSK systems, and the uniqueness of it for N in coherent CSK systems are due to the 'per bit' standardisation. If we ignore the transmission cost, any accuracy can be achieved by increasing N.

4. Comparison of Bit Error Rates of the Optimal and Correlation Decoders in Non-coherent Systems

It has been seen that the correlation decoder is not optimal in non-coherent systems and the optimal decoder will give a lower bit error rate than the correlation decoder. On the other hand, the optimal decoder requires much time to calculate, so it is of interest to know whether the optimal decoder gives a worthwhile improvement relative to cost. To assess this, we should compare exact bit error rates of the optimal decoder and the correlation decoder. However it seems hardly possible to calculate the exact bit error rate of the optimal decoder because of the complexity of the modulating factor. The calculation has been done successfully in only a few very simple case. Therefore simulated bit error rates of the optimal decoder ($\widehat{BER}_{opt}$) will be used instead, and compared with the exact bit error rates of the correlation decoder (BER_{cor}), by means of a standard statistical test. Denoting the exact bit error rate of the optimal decoder as BER_{opt}, the null hypothesis

$$H_0 : \quad BER_{opt} = BER_{cor},$$

is tested against the alternative hypothesis

$$H_1 : \quad BER_{opt} < BER_{cor}.$$

If the null hypothesis is not rejected, even though it should be theoretically, it might be said that there is no point practically in using the optimal decoder.

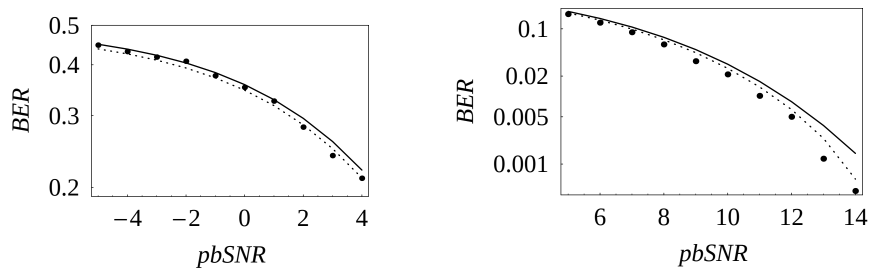
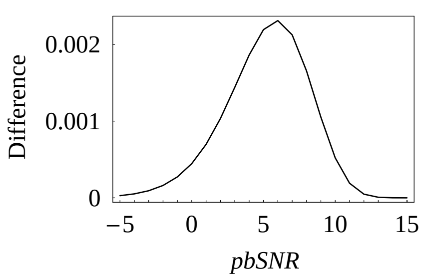
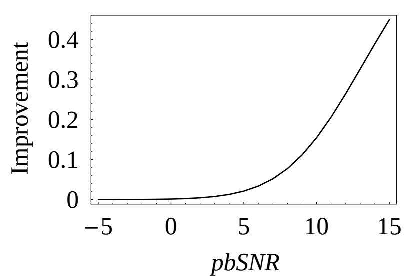


Figure 3. Bit error rates of the correlation decoder (solid) and simulated bit error rates of the optimal decoder based on 5000 samples (points) plotted against per bit signal to noise ratio *pbSNR* (left: $-5 \leq pbSNR \leq 4$, right: $5 \leq pbSNR \leq 14$,) in the case of non-coherent logistic spreading with spreading factor $N = 5$. Dotted line indicates the boundary of a critical region with a significance level 0.05.





Left: **Figure 4a**. The difference between exact bit error rates of the optimal decoder and the correlation decoder plotted against per bit signal to noise ratio *pbSNR* in the case of non-coherent balanced binary spreading with spreading factor $N = 2$.
Right: **Figure 4b**. The improvement of the optimal decoder in bit error rates plotted against per bit signal to noise ratio *pbSNR* in the case of non-coherent balanced binary spreading with spreading factor $N = 2$. The improvement is measured as $(BER_{cor} - BER_{opt})/BER_{cor}$.

Figure 3 shows that the null hypothesis can not be rejected at a small per bit signal to noise ratio while it is at a large per bit signal to noise ratio. It thus might be suggested that the optimal decoder does give a lower bit error rate than the correlation decoder for large *pbSNR*, and the larger *pbSNR*, the larger the improvement in bit error rates. Figure 4a and Figure 4b, in which the difference between BER_{cor} and BER_{opt}, and the improvement of BER_{opt} are plotted respectively, substantiates the above suggestion strongly. Considering the cost of the optimal decoder, however, it is doubtful if such improvement is large enough to encourage the use of the optimal decoder in practical situations. In the authors' simulation it took about 250 times more time on average to decode by the optimal decoder than the correlation decoder, and the difference seems to increase exponentially as the spreading factor N increases. It is clear that quick response is one of the most important aspects in communication systems, so the use of the optimal decoder might not be recommended straightforwardly and further investigation from both the points of view of statistics and engineering will be required.

5. **Conclusions and Discussion**

It has been seen in the paper that exact bit error rate calculations are sometimes possible and then preferable to simulations; correlation decoders are not optimal in non-coherent CSK systems in terms of maximum likelihood estimates of bit types, but are calculated more easily than optimal decoders and sufficiently useful in practical situations. However it should be noted that the importance of the optimal decoder is not weakened because the paper exemplified only CSK systems. It should also be noted that earlier Gaussian central limit theorem assumptions ignore chaotic dynamics and lead to inexact results which are only lower bounds on the error probabilities.

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